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Projective modules of finite type and monopoles over S^2

Giovanni Landi [a,b]

[a] *Dipartimento di Scienze Matematiche, Università di Trieste, P.le Europa 1, I-34127 Trieste, Italy*
[b] *INFN, Sezione di Napoli, Napoli, Italy*

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Dedicated to Jacopo

Abstract

We present a unifying description of all inequivalent vector bundles over the two-dimensional sphere S^2 by constructing suitable global projectors p via equivariant maps. Each projector determines the projective module of finite type of sections of the corresponding complex rank 1 vector bundle over S^2. The canonical connection $\nabla = p \circ d$ is used to compute the topological charges. Transposed projectors give opposite values for the charges, thus showing that transposition of projectors, although an isomorphism in K-theory, is not the identity map. Also, we construct the partial isometry yielding the equivalence between the tangent projector (which is trivial in K-theory) and the real form of the charge 2 projector.

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1. Preliminaries and introduction

Since the creation of noncommutative geometry, finite (meaning of finite type) projective modules as substitutes for vector bundles [1,3] are increasingly being used among (mathematical)-physicists. This substitution is based on the Serre–Swan's theorem [2,15] which constructs a complete equivalence between the category of (smooth) vector bundles over a (smooth) compact manifold M and bundle maps, and the category of finite projective modules over the commutative algebra $C(M)$ of (smooth) functions over M and module morphisms. The space $\Gamma(M, E)$ of smooth sections of a vector bundle $E \to M$ over a

E-mail address: landi@mathsun1.univ.trieste.it (G. Landi).

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compact manifold M is a finite projective module over the commutative algebra $C(M)$ and every finite projective $C(M)$-module can be realized as the module of sections of some vector bundle over M. The correspondence was already used in [9] to give an algebraic version of classical geometry, notably of the notions of connection and covariant derivative. But it has been with the advent of noncommutative geometry that the equivalence has received a new emphasis and has been used, among several other things, to generalize the concept of vector bundles to noncommutative geometry and to construct noncommutative gauge and gravity theories.

In this paper we present a finite-projective-module description of all monopoles configurations on the two-dimensional sphere S^2. This will be done by constructing a suitable global projector $p \in \mathbb{M}_{|n|+1}(C(S^2))$ which determines the module of sections of the vector bundles on which monopoles live, as the image of p in the trivial module $C(S^2)^{|n|+1}$ (corresponding to the trivial rank $(|n|+1)$-vector bundle over S^2). The integer $n \in \mathbb{Z}$, which characterizes the bundle, is the value of the topological charge of the monopole (i.e. the first Chern number of the corresponding bundle).

Now, a local expression for projectors corresponding to monopoles was given in [14]. Our presentation is a global one which does not use any local chart or partition of unity. The price we pay for this is that the projector carrying charge n is a matrix of dimension $(|n|+1) \times (|n|+1)$ while in [14] the projectors were always 2×2 matrices. Furthermore, our construction is based on a unifying description in terms of global equivariant maps. We express the projectors in terms of a more fundamental object, a vector-valued function of basic equivariant maps. In a sense, we may say that we '*deconstruct*' the projectors [11]. We also mention that projector's fields occur in the context of CP^n-models (see for instance [18]). The equations of motion of these models describe harmonic maps from S^2 into CP^n and there exists 'two-dimensional instantons' (configurations for which the action reaches a global minimum and solving duality equations); the corresponding line bundles can be described by means of projectors like the ones we present here.

The present construction is generalized to supergeometry in [12] where we report on a construction of 'graded monopoles' on the supersphere $S^{2,2}$. A friendly approach to modules of several kind (including finite projective) is in [10]. In the following we shall avoid writing explicitly the exterior product symbol for forms.

2. The general construction

Let us start by briefly describing the general scheme that will be given in details in the following sections. Let $\pi : S^3 \to S^2$ be the Hopf principal fibration over the sphere S^2 with $U(1)$ as structure group. We shall denote with $\mathcal{B}_{\mathbb{C}} =: C^\infty(S^3, \mathbb{C})$ the algebra of $\mathbb{C}$-valued smooth functions on the total space S^3 while $\mathcal{A}_{\mathbb{C}} =: C^\infty(S^2, \mathbb{C})$ will be the algebra of $\mathbb{C}$-valued smooth functions on the base space S^2. The algebra $\mathcal{A}_{\mathbb{C}}$ will not be distinguished from its image in the algebra $\mathcal{B}_{\mathbb{C}}$ via pull-back.

On $\mathbb{C}$ there are left actions of the group $U(1)$ and they are labeled by an integer $n \in \mathbb{Z}$, two representations corresponding to different integers being inequivalent. Let $C^\infty_{(n)}(S^3, \mathbb{C})$

be the collection of corresponding equivariant maps:

$$\varphi : S^3 \to \mathbb{C}, \qquad \varphi(p \cdot w) = w^{-n} \cdot \varphi(p), \tag{2.1}$$

with $\varphi \in C^\infty_{(n)}(S^3, \mathbb{C})$ and for any $p \in S^3$, $w \in U(1)$. The space $C^\infty_{(n)}(S^3, \mathbb{C})$ is a right module over the (pull-back of the) algebra $\mathcal{A}_\mathbb{C}$. Moreover, it is well known (see for instance [17]) that there is a module isomorphism between $C^\infty_{(n)}(S^3, \mathbb{C})$ and the (right) $\mathcal{A}_\mathbb{C}$-module of sections $\Gamma^\infty(S^2, E^{(n)})$ of the associated vector bundle $E^{(n)} = S^3 \times_{U(1)} \mathbb{C}$ over S^2. In the spirit of Serre–Swan's theorem [15], the module $\Gamma^\infty(S^2, E^{(n)})$ will be identified with the image in the trivial, rank N, module $(\mathcal{A}_\mathbb{C})^N$ of a projector $p \in \mathbb{M}_N(\mathcal{A}_\mathbb{C})$, the latter being the algebra of $N \times N$ matrices with entries in $\mathcal{A}_\mathbb{C}$, i.e. $\Gamma^\infty(S^2, E^{(n)}) = p(\mathcal{A}_\mathbb{C})^N$. The integer N will turn out to be given by

$$N = |n| + 1. \tag{2.2}$$

The bundle and the associated projector being of rank 1 (over $\mathbb{C}$), the projector will be written as a ket–bra valued function

$$p = |\psi\rangle\langle\psi|, \tag{2.3}$$

with

$$\langle\psi| = (\psi_1, \ldots, \psi_N), \tag{2.4}$$

a specific vector-valued function on S^3, thus a specific element of $(\mathcal{B}_\mathbb{C})^N$, the components being functions $\psi_j \in \mathcal{B}_\mathbb{C}$, $j = 1, \ldots, N$. The vector-valued function (2.4) will be normalized

$$\langle\psi|\psi\rangle = 1, \tag{2.5}$$

a fact implying that p is a projector

$$p^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = p, \quad p^\dagger = p, \tag{2.6}$$

with the symbol "†" denoting the adjoint. Furthermore, the normalization will also imply that p is of rank 1 over $\mathbb{C}$ because

$$\mathrm{tr}(p) = \langle\psi|\psi\rangle = 1. \tag{2.7}$$

In fact, the right end side of (2.7) is not the number 1 but rather the constant function 1. Then a normalized integration yields the number 1 as the value for the rank of the projector and of the associated vector bundle.

The transformation rule of the vector-valued function $|\psi\rangle$ under the right action of an element $w \in U(1)$ will be such that the projector p is invariant. Thus, its entries are functions on the base space S^2, i.e. they are elements of the algebra $\mathcal{A}_\mathbb{C}$ and $p \in \mathbb{M}_N(\mathcal{A}_\mathbb{C})$, as it should be. Elements of $(\mathcal{A}_\mathbb{C})^N$ will be denoted by the symbol

$$||f\rangle\rangle = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}, \tag{2.8}$$

with $f_1, \ldots, f_N$, elements of $\mathcal{A}_{\mathbb{C}}$. Then, the module isomorphism between sections and equivariant maps will be explicitly given by

$$\Gamma^\infty(S^2, E^{(n)}) \leftrightarrow C^\infty_{(n)}(S^3, \mathbb{C}),$$
$$\sigma = p\,||f\rangle\rangle \leftrightarrow \varphi^\sigma =: \langle\psi|\sigma\rangle = \langle\psi|\,|f\rangle\rangle = \sum_{j=1}^{N} \psi_j f_j, \tag{2.9}$$

where we have used the explicit identification $\Gamma^\infty(S^2, E^{(n)}) = p(\mathcal{A}_{\mathbb{C}})^N$, $N = |n| + 1$.

Having the projector, we can define a canonical connection (the Grassmann connection) on the module of sections by

$$\nabla =: p \circ d : \Gamma^\infty(S^2, E^{(n)}) \to \Gamma^\infty(S^2, E^{(n)}) \otimes_{\mathcal{A}_{\mathbb{C}}} \Omega^1(S^2, \mathbb{C}),$$
$$\nabla\sigma =: \nabla(p||f\rangle\rangle) = p(||\mathrm{d}f\rangle\rangle + \mathrm{d}p||f\rangle\rangle). \tag{2.10}$$

Its curvature $\nabla^2 : \Gamma^\infty(S^2, E^{(n)}) \to \Gamma^\infty(S^2, E^{(n)}) \otimes_{\mathcal{A}_{\mathbb{C}}} \Omega^2(S^2, \mathbb{C})$ is found to be

$$\nabla^2 = p(\mathrm{d}p)^2. \tag{2.11}$$

By means of a matrix trace, the first Chern class of the vector bundle is given as [6]

$$C_1(p) =: -\frac{1}{2\pi \mathrm{i}}\,\mathrm{tr}(\nabla^2) = -\frac{1}{2\pi \mathrm{i}}\,\mathrm{tr}(p(\mathrm{d}p)^2). \tag{2.12}$$

When integrated over S^2 it will yield the corresponding Chern number

$$c_1(p) = \int_{S^2} C_1(p). \tag{2.13}$$

As is shown in [1] these kind of formulae generalize directly in noncommutative differential geometry to provide analogues of Chern classes and topological numbers.

For rank 1 projectors of the form (2.3), the curvature is easily found to be given by

$$\nabla^2 = p(\mathrm{d}p)^2 = |\psi\rangle\,\langle \mathrm{d}\psi|\mathrm{d}\psi\rangle\,\langle\psi| \tag{2.14}$$

and the associated Chern form and Chern number are then

$$C_1(p) = -\frac{1}{2\pi \mathrm{i}}\langle \mathrm{d}\psi|\mathrm{d}\psi\rangle, \qquad c_1(p) = -\frac{1}{2\pi \mathrm{i}}\int_{S^2}\langle \mathrm{d}\psi|\mathrm{d}\psi\rangle. \tag{2.15}$$

By taking the transpose [1] of the projector (2.3) we still get a projector

$$q =: p^{\mathrm{t}} = |\phi\rangle\,\langle\phi|, \tag{2.16}$$

with the transposed bra-valued functions given by

$$\langle\phi| =: (|\psi\rangle)^{\mathrm{t}} = \langle\bar\psi| = (\bar\psi_1, \ldots, \bar\psi_N). \tag{2.17}$$

[1] Since we are considering only self-adjoint idempotents, i.e. projectors, the transpose is the same as the complex conjugate.

Here q is a projector ($q^2 = q = q^\dagger$) of rank 1 (over $\mathbb{C}$) are both consequences of the normalization $\langle\phi|\phi\rangle = \langle\psi|\psi\rangle = 1$. But it turns out that the transposed projector is not equivalent to the starting one, the corresponding topological charges differing in sign, i.e.

$$c_1(p^{\mathrm{t}}) = -c_1(p). \tag{2.18}$$

As we shall see, the change in sign comes from the antisymmetry of the exterior product for forms. Thus, transposing of projectors yields an isomorphism in K-theory which is not the identity map.

Given the connection (2.10), the corresponding connection 1-form on the equivariant maps, $A_\nabla \in \mathrm{End}_{\mathcal{B}_\mathbb{C}}(C^\infty_{U(1)}(S^3, \mathbb{C})) \otimes_{\mathcal{B}_\mathbb{C}} \Omega^1(S^3, \mathbb{C})$, has a very simple expression in terms of the vector-valued function $|\psi\rangle$ [11], being just

$$A_\nabla = \langle\psi|\mathrm{d}\psi\rangle. \tag{2.19}$$

The associated covariant derivative is given by

$$\nabla\varphi^\sigma =: \langle\psi|\nabla\sigma\rangle = (\mathrm{d} + \langle\psi|\mathrm{d}\psi\rangle)\varphi^\sigma \tag{2.20}$$

for any $\sigma \in \Gamma^\infty(S^2, E^{(n)})$ and where we have used the isomorphism (2.9). The connection form (2.19) is anti-Hermitian, a consequence of the normalization $\langle\psi|\psi\rangle = 1$

$$(A_\nabla)^\dagger =: \langle\mathrm{d}\psi|\psi\rangle = -\langle\psi|\mathrm{d}\psi\rangle = -A_\nabla. \tag{2.21}$$

Finally, on the ket-valued function $|\psi\rangle$ there will also be a global *left* action of the unitary group $SU(N) = \{s | ss^\dagger = 1\}$ which preserves the normalization

$$|\psi\rangle \mapsto |\psi^s\rangle = s|\psi\rangle, \qquad \langle\psi^s|\psi^s\rangle = 1. \tag{2.22}$$

The corresponding transformed projector

$$p^s = |\psi^s\rangle\langle\psi^s| = s|\psi\rangle\langle\psi|s^\dagger = sps^\dagger \tag{2.23}$$

is clearly equivalent to the starting one, the partial isometry being $v = sp$; indeed, one finds $vv^\dagger = p^s$ and $v^\dagger v = p$. Furthermore, the connection 1-form is left invariant

$$A_{\nabla^s} = \langle\psi^s|\mathrm{d}\psi^s\rangle = \langle\psi|s^\dagger s|\mathrm{d}\psi\rangle = A_\nabla. \tag{2.24}$$

To obtain new (in general gauge non-equivalent) connections one should act with group elements which do not preserve the normalization. Thus, let $g \in GL(N; \mathbb{C})$ act on the ket-valued function $|\psi\rangle$ by

$$|\psi\rangle \mapsto |\psi^g\rangle = [\langle\psi|g^\dagger g|\psi\rangle]^{-1/2} g|\psi\rangle. \tag{2.25}$$

The corresponding transformed projector

$$p^g = |\psi^g\rangle\langle\psi^g| = \langle\psi|g^\dagger g|\psi\rangle^{-1} g|\psi\rangle\langle\psi|g^\dagger = \langle\psi|g^\dagger g|\psi\rangle^{-1} gpg^\dagger \tag{2.26}$$

is again equivalent to the starting one, the partial isometry being now

$$v = [\langle\psi|g^\dagger g|\psi\rangle]^{-1/2} gp. \tag{2.27}$$

Indeed

$$\begin{aligned} vv^\dagger &= \langle\psi|g^\dagger g|\psi\rangle^{-1} gpg^\dagger = p^g, \\ v^\dagger v &= \langle\psi|g^\dagger g|\psi\rangle^{-1} pg^\dagger gp = \langle\psi|g^\dagger g|\psi\rangle^{-1} |\psi\rangle\,\langle\psi|g^\dagger g|\psi\rangle\,\langle\psi| = p. \end{aligned} \tag{2.28}$$

The associated connection 1-form is readily found to be

$$A_{\nabla^g} =: \langle\psi^g|\mathrm{d}\psi^g\rangle = \tfrac{1}{2}\langle\psi|g^\dagger g|\psi\rangle^{-1}[\langle\psi|g^\dagger g|\mathrm{d}\psi\rangle - \langle\mathrm{d}\psi|g^\dagger g|\psi\rangle]. \tag{2.29}$$

Thus, if $g \in SU(N)$, we get back the previous invariance of connections (2.24), while for $g \in GL(N)$ modulo $SU(N)$ we get new, gauge non-equivalent connections on the complex line bundle over S^2 determined by the projector p^g, line bundle which is (stable) isomorphic to the one determined by the projector p.

We conclude this section by mentioning that the use of equivariant mappings in the description of line bundles over S^2 (when thought of as the complex projective line CP^1) is known in algebraic geometry [7]. This is due to the fact that the sheaves corresponding to line bundles over CP^1 are sheaves of homogeneous functions.

3. The Hopf fibration over S^2

The $U(1)$ principal fibration $\pi : S^3 \to S^2$ over the two-dimensional sphere is explicitly realized as follows. The total space is the three-dimensional sphere

$$S^3 = \{(z_0, z_1) \in \mathbb{C}^2,\ \ |z_0|^2 + |z_1|^2 = 1\}, \tag{3.1}$$

with right $U(1)$-action

$$S^3 \times U(1) \to S^3, \qquad (z_0, z_1)\cdot w = (z_0 w, z_1 w). \tag{3.2}$$

Clearly $|z_0 w|^2 + |z_1 w|^2 = |z_0|^2 + |z_1|^2 = 1$. The bundle projection $\pi : S^3 \to S^2$ is just the Hopf projection and it is given by $\pi(z_0, z_1) =: (x_1, x_2, x_3)$

$$\begin{aligned} x_1 &= z_0\bar{z}_1 + z_1\bar{z}_0, \qquad x_2 = \mathrm{i}(z_0\bar{z}_1 - z_1\bar{z}_0), \\ x_3 &= |z_0|^2 - |z_1|^2 = -1 + 2|z_0|^2 = 1 - 2|z_1|^2 \end{aligned} \tag{3.3}$$

and one checks that $\sum_{\mu=1}^{3}(x_\mu)^2 = (|z_0|^2 + |z_1|^2)^2 = 1$. The inversion of (3.3) gives the basic ($\mathbb{C}$-valued) invariant functions on S^3

$$|z_0|^2 = \tfrac{1}{2}(1 + x_3), \qquad |z_1|^2 = \tfrac{1}{2}(1 - x_3), \qquad z_0\bar{z}_1 = \tfrac{1}{2}(x_1 - \mathrm{i}x_2), \tag{3.4}$$

a generic invariant (polynomial) function on S^3 being any function of the previous variables. Later on we shall also need the volume form of S^2 which turns out to be

$$\mathrm{d}(\mathrm{vol}(S^2)) = x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 + x_2\,\mathrm{d}x_3\,\mathrm{d}x_1 + x_3\,\mathrm{d}x_1\,\mathrm{d}x_2 = 2\mathrm{i}(\mathrm{d}z_0\,\mathrm{d}\bar{z}_0 + \mathrm{d}z_1\,\mathrm{d}\bar{z}_1). \tag{3.5}$$

3.1. The equivariant maps

Irreducible representations of the group $U(1)$ are labeled by an integer $n \in \mathbb{Z}$, any two representations associated with different integers being inequivalent. They can be explicitly given as left representations on $\mathbb{C}$

$$\rho_n : U(1) \times \mathbb{C} \to \mathbb{C}, \qquad (w, c) \mapsto \rho_n(w) \cdot c =: w^n c. \tag{3.6}$$

In order to construct the corresponding equivariant maps $\varphi : S^3 \to \mathbb{C}$ we shall distinguish between the two cases for which the integer n is negative or positive. In fact, from now on, we shall take the integer n to be always positive and consider the two cases corresponding to $\mp n$.

3.1.1. The equivariant maps for negative labels

Given any positive integer $n \in \mathbb{N}$, the generic equivariant map $\varphi_{-n} : S^3 \to \mathbb{C}$ corresponding to the representation of $U(1)$ labeled by $-n$ is of the form

$$\varphi_{-n}(z_0, z_1) = \sum_{k=0}^{n} (z_0)^{n-k} (z_1)^k g_k(z_0, z_1), \tag{3.7}$$

with g_k, $k = 0, 1, \ldots, n$, generic $\mathbb{C}$-valued functions on S^3 which are invariant under the right action of $U(1)$. Indeed

$$\begin{aligned} \varphi_{-n}((z_0, z_1)w) &= \sum_{k=0}^{n} (z_0 w)^{n-k} (z_1 w)^k g_k(z_0 w, z_1 w) \\ &= w^n \sum_{k=0}^{n} (z_0)^{n-k} (z_1)^k g_k(z_0, z_1) = \rho_{-n}(w)^{-1} \cdot \varphi_{-n}(z_0, z_1). \end{aligned} \tag{3.8}$$

We shall think of the functions g_k's as $\mathbb{C}$-valued functions on the base space S^2, namely as elements of the algebra $\mathcal{A}_{\mathbb{C}}$. The collection $C^\infty_{(-n)}(S^3, \mathbb{C})$ of equivariant maps is a right module over the (pull-back of) functions $\mathcal{A}_{\mathbb{C}}$.

3.1.2. The equivariant maps for positive labels

Given any positive integer $n \in \mathbb{N}$, the generic equivariant map $\varphi_n : S^3 \to \mathbb{C}$ corresponding to the representation of $U(1)$ labeled by n is of the form

$$\varphi_n(z_0, z_1) = \sum_{k=0}^{n} (\bar{z}_0)^{n-k} (\bar{z}_1)^k f_k(z_0, z_1), \tag{3.9}$$

with f_k, $k = 0, 1, \ldots, n$, generic $\mathbb{C}$-valued functions on S^3 which are invariant under the right action of $U(1)$. Indeed

$$\begin{aligned} \varphi_n((z_0, z_1)w) &= \sum_{k=0}^{n} (\overline{wz}_0)^{n-k} (\overline{wz}_1)^k f_k(z_0 w, z_1 w) \\ &= \bar{w}^n \sum_{k=0}^{n} (\bar{z}_0)^{n-k} (\bar{z}_1)^k f_k(z_0, z_1) = \rho_n(w)^{-1} \cdot \varphi_n(z_0, z_1). \end{aligned} \tag{3.10}$$

As before, we shall think of the functions f_k's as elements of the algebra $\mathcal{A}_{\mathbb{C}}$. And the collection $C^\infty_{(n)}(S^3, \mathbb{C})$ of equivariant maps will again be a right module over $\mathcal{A}_{\mathbb{C}}$.

3.2. The projectors and their charges

We are now ready to introduce the projectors. Again we shall take the integer n to be positive and keep separated the two cases corresponding to $\mp n$.

3.2.1. The construction of the projectors for negative labels

Given any positive integer $n \in \mathbb{N}$, consider the $(n+1)$-component vector-valued function on S^3 given by

$$\langle \psi_{-n}| =: \left((z_0)^n, \ldots, \begin{pmatrix} n \\ k \end{pmatrix}^{1/2} (z_0)^{n-k}(z_1)^k, \ldots, (z_1)^n \right), \tag{3.11}$$

where

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}, \quad k = 0, 1, \ldots, n \tag{3.12}$$

are the binomial coefficients. The vector-valued function (3.11) is normalized

$$\langle \psi_{-n}|\psi_{-n}\rangle = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (z_0)^{n-k}(z_1)^k(\bar{z}_0)^{n-k}(\bar{z}_1)^k = (|z_0|^2 + |z_1|^2)^n = 1. \tag{3.13}$$

Then, we can construct a projector in $\mathbb{M}_{n+1}(\mathcal{A}_{\mathbb{C}})$ by

$$p_{-n} =: |\psi_{-n}\rangle \langle \psi_{-n}|. \tag{3.14}$$

It is clear that p_{-n} is a projector

$$p_{-n}^2 =: |\psi_{-n}\rangle \langle \psi_{-n}|\psi_{-n}\rangle \langle \psi_{-n}| = |\psi_{-n}\rangle \langle \psi_{-n}| = p_{-n}, \qquad p_{-n}^\dagger = p_{-n}. \tag{3.15}$$

Moreover, it is of rank 1 because its trace is the constant function 1

$$\mathrm{tr}\, p_{-n} = \langle \psi_{-n}|\psi_{-n}\rangle = 1. \tag{3.16}$$

The $U(1)$-action (3.2) will transform the vector (3.11) multiplicatively

$$\langle \psi_{-n}| \mapsto \langle (\psi_{-n})^w| = w^n \langle \psi_{-n}| \quad \forall w \in U(1). \tag{3.17}$$

As a consequence the projector p_{-n} is invariant

$$p_{-n} \mapsto (p_{-n})^w = |(\psi_{-n})^w\rangle \langle (\psi_{-n})^w| = |\psi_{-n}\rangle \bar{w}^n w^n \langle \psi_{-n}| = |\psi_{-n}\rangle \langle \psi_{-n}| = p_{-n} \tag{3.18}$$

(being $\bar{w}w = 1$), and its entries are functions on the base space S^2, i.e. they are elements of $\mathcal{A}_{\mathbb{C}}$ as it should be. Thus, the right module of sections $\Gamma^\infty(S^2, E^{(-n)})$ of the associated bundle is identified with the image of p_{-n} in the trivial rank $n+1$ module $(\mathcal{A}_{\mathbb{C}})^{n+1}$ and the module isomorphism between sections and equivariant maps is given by

$$\Gamma^{\infty}(S^2, E^{(-n)}) \leftrightarrow C^{\infty}_{(-n)}(S^3, \mathbb{C}), \qquad \sigma = p_{-n}|g\rangle\rangle \leftrightarrow \varphi^{\sigma}_{-n} = \langle \psi_{-n}|\sigma\rangle,$$

$$\varphi^{\sigma}_{-n}(z_0, z_1) = \langle \psi_{-n}| \begin{pmatrix} g_0 \\ \vdots \\ g_n \end{pmatrix} = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}^{1/2} (z_0)^{n-k}(z_1)^k g_k(z_0, z_1), \tag{3.19}$$

with $g_0, \ldots, g_n$ generic elements in $\mathcal{A}_{\mathbb{C}}$. By comparison with (3.7) it is obvious that the previous map is a module isomorphism, the extra factors $\begin{pmatrix} n \\ k \end{pmatrix}^{1/2}$ being inessential to this purpose since they could be absorbed in a redefinition of the functions.

The canonical connection associated with the projector p_{-n}

$$\nabla = p_{-n} \circ \mathrm{d} : \Gamma^{\infty}(S^2, E^{(-n)}) \to \Gamma^{\infty}(S^2, E^{(-n)}) \otimes_{\mathcal{A}_{\mathbb{C}}} \Omega^1(S^2, \mathbb{C}) \tag{3.20}$$

has curvature given by

$$\nabla^2 = p_{-n}(\mathrm{d}p_{-n})^2 = |\psi_{-n}\rangle \langle \mathrm{d}\psi_{-n}|\mathrm{d}\psi_{-n}\rangle \langle \psi_{-n}|. \tag{3.21}$$

The corresponding Chern number is

$$c_1(p_{-n}) =: -\frac{1}{2\pi \mathrm{i}}\int_{S^2} \mathrm{tr}(p_{-n}(\mathrm{d}p_{-n})^2) = -\frac{1}{2\pi \mathrm{i}}\int_{S^2} \langle \mathrm{d}\psi_{-n}|\mathrm{d}\psi_{-n}\rangle. \tag{3.22}$$

Now, a lengthy but straightforward computation shows that

$$\langle \mathrm{d}\psi_{-n}|\mathrm{d}\psi_{-n}\rangle = n(\mathrm{d}z_0\, \mathrm{d}\bar{z}_0 + \mathrm{d}z_1\, \mathrm{d}\bar{z}_1) = \frac{n}{2\mathrm{i}}\mathrm{d}(\mathrm{vol}(S^2)), \tag{3.23}$$

which, when substituted in (3.22) gives

$$c_1(p_{-n}) = \frac{n}{4\pi}\int_{S^2} \mathrm{d}(\mathrm{vol}(S^2)) = \frac{n}{4\pi}4\pi = n. \tag{3.24}$$

3.2.2. The construction of the projectors for positive labels

Given any positive integer $n \in \mathbb{N}$, consider the $(n+1)$-component vector-valued function on S^3 given by

$$\langle \psi_n| =: \left((\bar{z}_0)^n, \ldots, \begin{pmatrix} n \\ k \end{pmatrix}^{1/2} (\bar{z}_0)^{n-k}(\bar{z}_1)^k, \ldots, (\bar{z}_1)^n \right), \tag{3.25}$$

it is normalized

$$\langle \psi_n|\psi_n\rangle = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (\bar{z}_0)^{n-k}(\bar{z}_1)^k (z_0)^{n-k}(z_1)^k = (|z_0|^2 + |z_1|^2)^n = 1. \tag{3.26}$$

As before, a projector in $\mathbb{M}_{n+1}(\mathcal{A}_{\mathbb{C}})$ is constructed by

$$p_n =: |\psi_n\rangle \langle \psi_n|. \tag{3.27}$$

Indeed $p_n^2 =: |\psi_n\rangle\langle\psi_n|\psi_n\rangle\langle\psi_n| = |\psi_n\rangle\langle\psi_n| = p_n$ and $p_n^\dagger = p_n$. Furthermore, the projector p_n is of rank 1 because its trace is the constant function 1, tr $p_n = \langle\psi_n|\psi_n\rangle = 1$.

The $U(1)$-action (3.2) will now transform the vector (3.25) by

$$\langle\psi_n| \mapsto \langle(\psi_n)^w| = \bar{w}^n\langle\psi_n| \quad \forall w \in U(1). \tag{3.28}$$

As a consequence, the projector p_n is invariant

$$p_n \mapsto (p_n)^w = |(\psi_n)^w\rangle\langle(\psi_n)^w| = |\psi_n\rangle w^n\bar{w}^n\langle\psi_n| = |\psi_n\rangle\langle\psi_n| = p_n \tag{3.29}$$

(being $w\bar{w} = 1$), and its entries are again functions on the base space S^2, i.e. they are elements of $\mathcal{A}_{\mathbb{C}}$. Thus, also the right module of sections $\Gamma^\infty(S^2, E^{(n)})$ is identified with the image of p_n in the trivial rank $n+1$ module $(\mathcal{A}_{\mathbb{C}})^{n+1}$. The module isomorphism between sections and equivariant maps is now given by

$$\Gamma^\infty(S^2, E^{(n)}) \leftrightarrow C^\infty_{(n)}(S^3, \mathbb{C}), \qquad \sigma = p_n||f\rangle\rangle \leftrightarrow \varphi_n^\sigma = \langle\psi_n|\sigma\rangle,$$

$$\varphi_n^\sigma(z_0, z_1) = \langle\psi_n|\begin{pmatrix} f_0 \\ \vdots \\ f_n \end{pmatrix} = \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}^{1/2}(\bar{z}_0)^{n-k}(\bar{z}_1)^k f_k(z_0, z_1), \tag{3.30}$$

with $f_0, \dots, f_n$ generic elements in $\mathcal{A}_{\mathbb{C}}$. By comparison with (3.9) it is obvious that the previous map is a module isomorphism (again the extra factors could be absorbed in a redefinition of the functions).

The canonical connection associated with the projector p_n

$$\nabla = p_n \circ d : \Gamma^\infty(S^2, E^{(n)}) \to \Gamma^\infty(S^2, E^{(n)}) \otimes_{\mathcal{A}_{\mathbb{C}}} \Omega^1(S^2, \mathbb{C}) \tag{3.31}$$

has a curvature given by $\nabla^2 = p_n(\mathrm{d}p_n)^2 = |\psi_n\rangle\langle\mathrm{d}\psi_n|\mathrm{d}\psi_n\rangle\langle\psi_n|$, and the corresponding Chern number is

$$c_1(p_n) =: -\frac{1}{2\pi\mathrm{i}}\int_{S^2}\mathrm{tr}(p_n(\mathrm{d}p_n)^2) = -\frac{1}{2\pi\mathrm{i}}\int_{S^2}\langle\mathrm{d}\psi_n|\mathrm{d}\psi_n\rangle. \tag{3.32}$$

Now, the vector-valued functions $\langle\psi_n|$ and $\langle\psi_{-n}|$ transform one into the other by the exchange $z_0 \leftrightarrow \bar{z}_0$ and $z_1 \leftrightarrow \bar{z}_1$. Thus, as a consequence of the antisymmetry of the wedge product for 1-forms one has

$$\langle\mathrm{d}\psi_n|\mathrm{d}\psi_n\rangle = -\langle\mathrm{d}\psi_{-n}|\mathrm{d}\psi_{-n}\rangle = -n(\mathrm{d}z_0\,\mathrm{d}\bar{z}_0 + \mathrm{d}z_1\,\mathrm{d}\bar{z}_1) = -\frac{n}{2\mathrm{i}}\,\mathrm{d}(\mathrm{vol}(S^2)), \tag{3.33}$$

which, when substituted in (3.32) gives

$$c_1(p_n) = -\frac{n}{4\pi}\int_{S^2}\mathrm{d}(\mathrm{vol}(S^2)) = -\frac{n}{4\pi}4\pi = -n. \tag{3.34}$$

In fact, the functions $\langle\psi_n|$ and $\langle\psi_{-n}|$ are the one transposed of the other,[2] i.e.

$$\langle\psi_n| = (|\psi_{-n}\rangle)^{\mathrm{t}} = \langle\overline{\psi_{-n}}| \tag{3.35}$$

[2] As already remarked, in our case, transposition is the same as complex conjugation.

and the corresponding projectors are related by transposition

$$p_n = (p_{-n})^{\mathrm{t}}. \tag{3.36}$$

Thus, by transposing a projector we get an inequivalent one. [3] This inequivalence is a manifestation of the fact that transposing of projectors yields an isomorphism in the reduced K-theory group $\tilde{K}(S^2)$, which is not the identity map.

Example. Here we give the explicit projectors corresponding to the lowest values of the charges, ± 1, while in the next section we give the one corresponding to charge ± 2. By using the definition (3.3) for the coordinate functions on S^2, we find that

$$p_{-1} = \begin{pmatrix} |z|_0^2 & z_1\bar{z}_0 \\ z_0\bar{z}_1 & |z|_1^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+x_3 & x_1+\mathrm{i}x_2 \\ x_1-\mathrm{i}x_2 & 1-x_3 \end{pmatrix},$$

$$p_1 = \begin{pmatrix} |z|_0^2 & z_0\bar{z}_1 \\ z_1\bar{z}_0 & |z|_1^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+x_3 & x_1-\mathrm{i}x_2 \\ x_1+\mathrm{i}x_2 & 1-x_3 \end{pmatrix}. \tag{3.37}$$

It is evident that these projectors are one the transposed (or equivalently, the complex conjugate) of the other.

3.3. The monopole connections

We are now ready to construct explicitly the monopole connections. The connection 1-forms (2.19) associated with the projectors $p_{\mp n}$ are given by

$$A_{\mp n} = \langle \psi_{\mp n} | \mathrm{d}\psi_{\mp n} \rangle. \tag{3.38}$$

They are anti-Hermitian

$$(A_{\mp n})^{\dagger} = \langle \psi_{\mathrm{d}\mp n} | \psi_{\mp n} \rangle = -\langle \psi_{\mp n} | \mathrm{d}\psi_{\mp n} \rangle = -A_{\mp n}, \tag{3.39}$$

so they are valued in $\mathrm{i}\mathbb{R}$, the Lie algebra of $U(1)$. A straightforward computation yields

$$A_{\mp n} = \mp n(\bar{z}_0\,\mathrm{d}z_0 + \bar{z}_1\,\mathrm{d}z_1) = \mp n A_1, \tag{3.40}$$

with

$$A_1 = \bar{z}_0\,\mathrm{d}z_0 + \bar{z}_1\,\mathrm{d}z_1 \tag{3.41}$$

the charge -1 monopole connection form [13,17]. In [16] a local expression of the connection A_{-n} (correspondingly, we recall, to charge n) was given as the pull-back to S^2 of the Hodge form of the projective space CP^n. The pull-back is associated with an embedding of $S^2(= CP^1)$ into CP^n given by the vector-valued function $\langle \psi_{-n}|$ in (3.11) and with the functions z_0, z_1 thought of as homogeneous coordinates on CP^1 [7].

[3] Unless the projector is the identity.

Finally, the invariance (2.24) states the invariance of the connection 1-form (3.38) under the global left action of $SU(N)$. Gauge non-equivalent connections are obtained by the formula (2.29)

$$A^{g}_{\mp n} = \tfrac{1}{2}\langle\psi_{\mp n}|g^{\dagger}g|\psi_{\mp n}\rangle^{-1}[\langle\psi_{\mp n}|g^{\dagger}g|\mathrm{d}\psi_{\mp n}\rangle - \langle\mathrm{d}\psi_{\mp n}|g^{\dagger}g|\psi_{\mp n}\rangle], \tag{3.42}$$

with $g \in GL(N;\mathbb{C})$ modulo $SU(N)$ and $\langle\psi_{\mp n}|$ are given by (3.11) and (3.25), respectively.

A description of gauge theories which uses projector's fields as field variables has been suggested in [4,5]. To our knowledge there was no further work in this direction.

4. The tangent projector vs the charge 2 projector

It is well known that the bundle TS^2 tangent to S^2, although not trivial as a bundle, is trivial in real K-theory (stable triviality) [8]; it is a consequence of the fact that by adding to TS^2 the real rank 1 trivial bundle one gets the real rank 3 trivial bundle. On the other hand, it is also well known (see for instance [17]) that TS^2 can be identified with the real form of the complex charge 2 monopole bundle over S^2. This identification is an instance of the general result that equates the top Chern class of a complex vector bundle with the Euler class of the real form of the bundle. We shall prove this equivalence at the level of K-theory by constructing explicitly the partial isometry between the tangent projector and the real form of the charge 2 monopole projector. This will also show that classes which are not trivial in complex K-theory may become trivial when translated into real K-theory.

4.1. The tangent projector

We shall use real cartesian coordinates (x_1, x_2, x_3), $\sum_{\mu=1}^{3}(x_\mu)^2 = 1$, for the sphere S^2 and denote with $\mathcal{A}_{\mathbb{R}} =: C^{\infty}(S^3, \mathbb{R})$ the algebra of smooth real-valued functions on S^2. The (module of smooth sections of the) normal bundle over S^2 is realized as the image, in the trivial, real rank 3 module $(\mathcal{A}_{\mathbb{R}})^3$, of the *normal projector*

$$p_{\mathrm{nor}} = |\psi_{\mathrm{nor}}\rangle\,\langle\psi_{\mathrm{nor}}|, \qquad \langle\psi_{\mathrm{nor}}| = (x_1, x_2, x_3). \tag{4.1}$$

It is clear that p_{nor} is a projector, $p^2_{\mathrm{nor}} = p_{\mathrm{nor}} = p^{\dagger}_{\mathrm{nor}}$, of real rank 1. Moreover, p_{nor} is trivial since it admits a nonvanishing section.

Then, the (module of smooth sections of the) tangent bundle is simply realized as the image in $(\mathcal{A}_{\mathbb{R}})^3$ of the *tangent projector*

$$p_{\mathrm{tan}} = \mathbb{I} - p_{\mathrm{nor}} = \mathbb{I} - |\psi_{\mathrm{nor}}\rangle\,\langle\psi_{\mathrm{nor}}| = \begin{pmatrix} 1-(x_1)^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & 1-(x_2)^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & 1-(x_3)^2 \end{pmatrix}. \tag{4.2}$$

That the tangent bundle is of real rank 2 is translated in the fact that the trace of p_{tan} is equal to the constant function 2, $\mathrm{tr}(p_{\mathrm{tan}}) = 2$. The tangent bundle is stable trivial as well and its 'topological charge' vanishes. Indeed, by using the fact that

$$\langle\psi_{\rm nor}|{\rm d}\psi_{\rm nor}\rangle = x_1\,{\rm d}x_1 + x_2\,{\rm d}x_2 + x_3\,{\rm d}x_3 = 0, \tag{4.3}$$

it is straightforward to find

$$\begin{aligned} p_{\tan}({\rm d}p_{\tan})^2 &= |{\rm d}\psi_{\rm nor}\rangle\,\langle{\rm d}\psi_{\rm nor}| = {\rm d}x_1\,{\rm d}x_2 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad + {\rm d}x_2\,{\rm d}x_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + {\rm d}x_3\,{\rm d}x_1 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.4}$$

As a consequence

$$C_1(p_{\tan}) =: -\frac{1}{2\pi{\rm i}}{\rm tr}(p_{\tan}({\rm d}p_{\tan})^2) = -\frac{1}{2\pi{\rm i}}{\rm tr}(|{\rm d}\psi_{\rm nor}\rangle\,\langle{\rm d}\psi_{\rm nor}|) = 0. \tag{4.5}$$

For later convenience, we need to express the tangent projector as a sum of three pieces. Let us then introduce the three vector fields on S^2 which generate the action of $SU(2)$ on S^2. They are given by

$$V_l = \sum_{j,k=0}^{3} \varepsilon_{ljk} x_j \frac{\partial}{\partial x_k}, \qquad l = 1, 2, 3 \tag{4.6}$$

and clearly they are not independent; indeed

$$\sum_{l=0}^{3} x_l V_l = 0. \tag{4.7}$$

We shall write the vector fields (4.6) as vector-valued functions on S^2

$$\langle V_1| = (0, -x_3, x_2), \qquad \langle V_2| = (x_3, 0, -x_1), \qquad \langle V_3| = (-x_2, x_1, 0). \tag{4.8}$$

Then, it is an easy computation to show that the tangent projector can be written as

$$p_{\tan} = |V_1\rangle\,\langle V_1| + |V_2\rangle\,\langle V_2| + |V_3\rangle\,\langle V_3|. \tag{4.9}$$

Finally, the following properties are easily established:

$$\begin{aligned} &\langle\psi_{\rm nor}|V_l\rangle = 0, \quad l = 1, 2, 3, \qquad p_{\tan}|V_l\rangle = |V_l\rangle, \quad l = 1, 2, 3, \\ &(p_{\tan})_{kl} = \langle V_k|V_l\rangle, \quad k, l = 1, 2, 3, \qquad {\rm tr}(p_{\tan}) = \sum_{l=0}^{3}\langle V_l|V_l\rangle = 2\sum_{\mu=1}^{3}(x_\mu)^2 = 2. \end{aligned} \tag{4.10}$$

4.2. The charge 2 projector and its real form

It turns out that in order to prove the equivalence we are looking for, it is best to 'change bases' for the equivariant functions and as a consequence to construct a charge 2 projector which is not the same as the one given by (3.14) but it is of course equivalent to it. Let us then consider the following vector-valued equivariant map on S^3

$$\langle\tilde{\psi}_{-2}| = \frac{1}{\sqrt{2}}((z_1)^2 - (z_0)^2, (z_1)^2 + (z_0)^2, 2z_0z_1) \tag{4.11}$$

(we recall that the label -2 characterizes the type of representation of $U(1)$ and that it corresponds to charge 2 as we shall also see presently). The vector-valued function (4.11) is normalized

$$\langle\tilde{\psi}_{-2}|\tilde{\psi}_{-2}\rangle = (|z|_0^2 + |z|_1^2)^2 = 1. \tag{4.12}$$

As a consequence, the following is a complex rank 1 projector in $\mathbb{M}_3(\mathcal{A}_{\mathbb{C}})$

$$\tilde{p}_{-2} =: |\tilde{\psi}_{-2}\rangle\,\langle\tilde{\psi}_{-2}|. \tag{4.13}$$

That the projector $\tilde{p}_{-2}$ is equivalent to the projector p_{-2} given by (3.14) for the value $n = 2$, is best seen by computing its topological charge. A simple computation shows that

$$\langle d\tilde{\psi}_{-2}|d\tilde{\psi}_{-2}\rangle = 2(dz_0\, d\bar{z}_0 + dz_1\, d\bar{z}_1) = \frac{1}{\mathrm{i}} d(\mathrm{vol}(S^2)), \tag{4.14}$$

which, in turn, gives

$$c_1(\tilde{p}_{-2}) = -\frac{1}{2\pi\mathrm{i}}\int_{S^2}\langle d\tilde{\psi}_{-2}|d\tilde{\psi}_{-2}\rangle = \frac{1}{2\pi}\int_{S^2} d(\mathrm{vol}(S^2)) = 2, \tag{4.15}$$

as it should be. This shows the equivalence between $\tilde{p}_{-2}$ and p_{-2} (of course one could also directly construct the corresponding partial isometry).

Next, we express the projector (4.13) in terms of the coordinate functions on S^2. It turns out that

$$\tilde{p}_{-2} = \frac{1}{2}\begin{pmatrix} 1-(x_1)^2 & -x_3 - \mathrm{i}x_1x_2 & -\mathrm{i}x_2 - x_1x_3 \\ -x_3 + \mathrm{i}x_1x_2 & 1-(x_2)^2 & x_1 + \mathrm{i}x_2x_3 \\ \mathrm{i}x_2 - x_1x_3 & x_1 - \mathrm{i}x_2x_3 & 1-(x_3)^2 \end{pmatrix}. \tag{4.16}$$

From the general considerations described before, the transpose of this projector would carry charge -2.

Let us now turn to real forms. The real form $(\tilde{p}_{-2})^{\mathbb{R}}$ of the projector $\tilde{p}_{-2}$ in (4.16) will be a projector in $\mathbb{M}_6(\mathcal{A}_{\mathbb{R}})$ and it is obtained by the substitution

$$a + \mathrm{i}b \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \forall a, b \in \mathbb{R}. \tag{4.17}$$

One finds that

$$(\tilde{p}_{-2})^{\mathbb{R}} = \frac{1}{2}\begin{pmatrix} 1-(x_1)^2 & 0 & -x_3 & x_1x_2 & -x_1x_3 & x_2 \\ 0 & 1-(x_1)^2 & -x_1x_2 & -x_3 & -x_2 & -x_1x_3 \\ -x_3 & -x_1x_2 & 1-(x_2)^2 & 0 & x_1 & -x_2x_3 \\ x_1x_2 & -x_3 & 0 & 1-(x_2)^2 & x_2x_3 & x_1 \\ -x_1x_3 & -x_2 & x_1 & x_2x_3 & 1-(x_3)^2 & 0 \\ x_2 & -x_1x_3 & -x_2x_3 & x_1 & 0 & 1-(x_3)^2 \end{pmatrix}. \tag{4.18}$$

That $(\tilde{p}_{-2})^{\mathbb{R}}$ is a projector will also be evident from the analysis of next section.

4.3. *The partial isometry between $p_{\tan}$ and $(\tilde{p}_{-2})^{\mathbb{R}}$*

The next step consists in expressing the projector $(\tilde{p}_{-2})^{\mathbb{R}}$ in (4.18) as a sum of three pieces, just as we have done for the tangent projector in (4.9). It turns out that

$$(\tilde{p}_{-2})^{\mathbb{R}} = |W_1\rangle\,\langle W_1| + |W_2\rangle\,\langle W_2| + |W_3\rangle\,\langle W_3|, \tag{4.19}$$

with

$$\begin{aligned} \langle W_1| &= \frac{1}{\sqrt{2}}(1-(x_1)^2 \quad 0 \quad -x_3 \quad x_1x_2 \quad -x_1x_3 \quad x_2), \\ \langle W_2| &= \frac{1}{\sqrt{2}}(-x_1x_2 \quad x_3 \quad 0 \quad -1+(x_2)^2 \quad -x_2x_3 \quad -x_1), \\ \langle W_3| &= \frac{1}{\sqrt{2}}(-x_1x_3 \quad -x_2 \quad x_1 \quad x_2x_3 \quad 1-(x_3)^2 \quad 0). \end{aligned} \tag{4.20}$$

These three vector-valued functions are not independent since

$$\sum_{l=0}^{3} x_l \langle W_l| = 0. \tag{4.21}$$

There is a simple relation between the three vector-valued functions W_l in (4.20) and the corresponding V_l in (4.6). Indeed

$$u|V_l\rangle = |W_l\rangle, \quad l = 1, 2, 3, \tag{4.22}$$

with the matrix-valued function u given by

$$u = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -x_3 & x_2 \\ 1-(x_1)^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & 1-(x_2)^2 & -x_2x_3 \\ -x_3 & 0 & x_1 \\ -x_2 & x_1 & 0 \\ -x_1x_3 & -x_2x_3 & 1-(x_3)^2 \end{pmatrix}. \tag{4.23}$$

The matrix u turns out to be the partial isometry we are looking for. A lengthy computation shows that

$$u^{\dagger}u = p_{\tan}, \qquad uu^{\dagger} = (\tilde{p}_{-2})^{\mathbb{R}}. \tag{4.24}$$

This proves the equivalence between the two projectors $p_{\tan}$ and $(\tilde{p}_{-2})^{\mathbb{R}}$ and finishes the K-theory version of the isomorphism between the tangent bundle TS^2 and the real form of the complex charge 2 monopole bundle over S^2.

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